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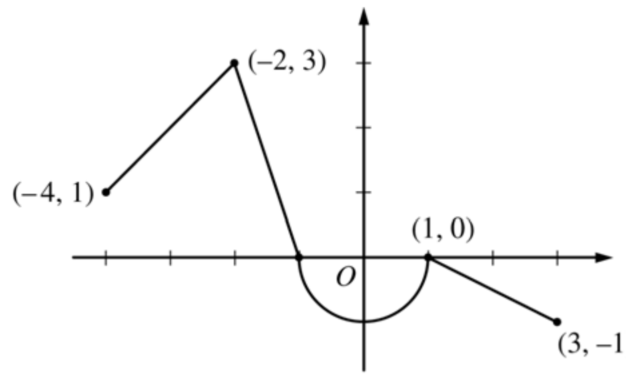
2012 SCORING GUIDELINES

Question 3

Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt.$



Graph of f

(a) Find the values of $g(2)$ and $g(-2)$.

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

(a) $g(2) = \int_1^2 f(t)\,dt = -\frac{1}{2}(1)\left(\frac{1}{2}\right) = -\frac{1}{4}$ $g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt$ $= -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{3}{2}$	$2 : \begin{cases} 1 : g(2) \\ 1 : g(-2) \end{cases}$
(b) $g'(x) = f(x) \Rightarrow g'(-3) = f(-3) = 2$ $g''(x) = f'(x) \Rightarrow g''(-3) = f'(-3) = 1$	$2 : \begin{cases} 1 : g'(-3) \\ 1 : g''(-3) \end{cases}$
(c) The graph of g has a horizontal tangent line where $g'(x) = f(x) = 0$. This occurs at $x = -1$ and $x = 1$. $g'(x)$ changes sign from positive to negative at $x = -1$. Therefore, g has a relative maximum at $x = -1$. $g'(x)$ does not change sign at $x = 1$. Therefore, g has neither a relative maximum nor a relative minimum at $x = 1$.	$3 : \begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : x = -1 \text{ and } x = 1 \\ 1 : \text{answers with justifications} \end{cases}$
(d) The graph of g has a point of inflection at each of $x = -2$, $x = 0$, and $x = 1$ because $g''(x) = f'(x)$ changes sign at each of these values.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{explanation} \end{cases}$

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Question 4

The function f is defined by $f(x) = \sqrt{25 - x^2}$ for $-5 \le x \le 5$.

(a) Find $f'(x)$.

(b) Write an equation for the line tangent to the graph of f at $x = -3$.

(c) Let g be the function defined by $g(x) = \begin{cases} f(x) & \text{for } -5 \le x \le -3 \\ x + 7 & \text{for } -3 < x \le 5. \end{cases}$

Is g continuous at $x = -3$? Use the definition of continuity to explain your answer.

(d) Find the value of $\int_0^5 x\sqrt{25 - x^2}\,dx$.

(a) $f'(x) = \frac{1}{2}\left(25 - x^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{25 - x^2}}, \quad -5 < x < 5$

2 : $f'(x)$

(b) $f'(-3) = \frac{3}{\sqrt{25 - 9}} = \frac{3}{4}$

$f(-3) = \sqrt{25 - 9} = 4$

An equation for the tangent line is $y = 4 + \frac{3}{4}(x + 3)$.

2 : $\begin{cases} 1 : f'(-3) \\ 1 : \text{answer} \end{cases}$

(c) $\lim_{x \to -3^-} g(x) = \lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} \sqrt{25 - x^2} = 4$

$\lim_{x \to -3^+} g(x) = \lim_{x \to -3^+} (x + 7) = 4$

Therefore, $\lim_{x \to -3} g(x) = 4$.

$g(-3) = f(-3) = 4$

So, $\lim_{x \to -3} g(x) = g(-3)$.

Therefore, g is continuous at $x = -3$.

2 : $\begin{cases} 1 : \text{considers one-sided limits} \\ 1 : \text{answer with explanation} \end{cases}$

(d) Let $u = 25 - x^2 \Rightarrow du = -2x\,dx$

$$\int_0^5 x\sqrt{25 - x^2}\,dx = -\frac{1}{2}\int_{25}^{0} \sqrt{u}\,du$$

$$= \left[-\frac{1}{2} \cdot \frac{2}{3}u^{3/2}\right]_{u=25}^{u=0}$$

$$= -\frac{1}{3}(0 - 125) = \frac{125}{3}$$

3 : $\begin{cases} 2 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

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Question 5

The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

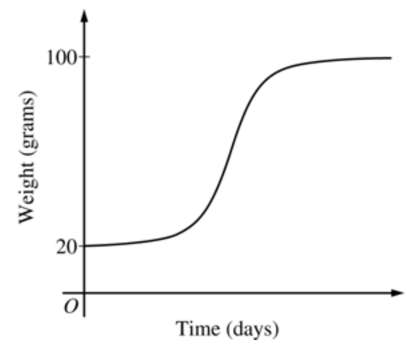
$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.

(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.



(a) $\left.\frac{dB}{dt}\right\|_{B=40} = \frac{1}{5}(60) = 12$ $\left.\frac{dB}{dt}\right\|_{B=70} = \frac{1}{5}(30) = 6$ Because $\left.\frac{dB}{dt}\right\|_{B=40} > \left.\frac{dB}{dt}\right\|_{B=70}$, the bird is gaining weight faster when it weighs 40 grams.	2 : { 1 : uses $\frac{dB}{dt}$ 1 : answer with reason }
(b) $\frac{d^2B}{dt^2} = -\frac{1}{5}\frac{dB}{dt} = -\frac{1}{5}\cdot\frac{1}{5}(100 - B) = -\frac{1}{25}(100 - B)$ Therefore, the graph of B is concave down for $20 \le B < 100$. A portion of the given graph is concave up.	2 : { 1 : $\frac{d^2B}{dt^2}$ in terms of B 1 : explanation }
(c) $\frac{dB}{dt} = \frac{1}{5}(100 - B)$ $\int \frac{1}{100 - B}\,dB = \int \frac{1}{5}\,dt$ $-\ln\|100 - B\| = \frac{1}{5}t + C$ Because $20 \le B < 100$, $\|100 - B\| = 100 - B$. $-\ln(100 - 20) = \frac{1}{5}(0) + C \Rightarrow -\ln(80) = C$ $100 - B = 80e^{-t/5}$ $B(t) = 100 - 80e^{-t/5}, \quad t \ge 0$	5 : { 1 : separation of variables 1 : antiderivatives 1 : constant of integration 1 : uses initial condition 1 : solves for B } Note: max $2/5$ [1-1-0-0-0] if no constant of integration Note: $0/5$ if no separation of variables

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Question 6

For $0 \le t \le 12$, a particle moves along the x-axis. The velocity of the particle at time t is given by $v(t) = \cos\left(\frac{\pi}{6}t\right)$. The particle is at position $x = -2$ at time $t = 0$.

(a) For $0 \le t \le 12$, when is the particle moving to the left?

(b) Write, but do not evaluate, an integral expression that gives the total distance traveled by the particle from time $t = 0$ to time $t = 6$.

(c) Find the acceleration of the particle at time t. Is the speed of the particle increasing, decreasing, or neither at time $t = 4$? Explain your reasoning.

(d) Find the position of the particle at time $t = 4$.

(a) $v(t) = \cos\left(\frac{\pi}{6}t\right) = 0 \;\Rightarrow\; t = 3,\ 9$

The particle is moving to the left when $v(t) < 0$.
This occurs when $3 < t < 9$.

$2 : \begin{cases} 1 : \text{considers } v(t) = 0 \\ 1 : \text{interval} \end{cases}$

(b) $\int_0^6 |v(t)|\, dt$

1 : answer

(c) $a(t) = -\frac{\pi}{6}\sin\left(\frac{\pi}{6}t\right)$

$a(4) = -\frac{\pi}{6}\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}\pi}{12} < 0$

$v(4) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} < 0$

The speed is increasing at time $t = 4$, because velocity and acceleration have the same sign.

$3 : \begin{cases} 1 : a(t) \\ 2 : \text{conclusion with reason} \end{cases}$

(d)
$$\begin{aligned} x(4) &= -2 + \int_0^4 \cos\left(\frac{\pi}{6}t\right) dt \\ &= -2 + \left[\frac{6}{\pi}\sin\left(\frac{\pi}{6}t\right)\right]_0^4 \\ &= -2 + \frac{6}{\pi}\left[\sin\left(\frac{2\pi}{3}\right) - 0\right] \\ &= -2 + \frac{6}{\pi}\cdot\frac{\sqrt{3}}{2} = -2 + \frac{3\sqrt{3}}{\pi} \end{aligned}$$

$3 : \begin{cases} 1 : \text{antiderivative} \\ 1 : \text{uses initial condition} \\ 1 : \text{answer} \end{cases}$