Question 3

(a) $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$

$\Rightarrow\ 7 = \int_{-6}^{-2} f(x)\,dx + 2 + \left(9 - \frac{9\pi}{4}\right)$

$\Rightarrow\ \int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$

3 : { 1 : $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$
1 : $\int_{-2}^{5} f(x)\,dx$
1 : answer

(b) $\int_{3}^{5} (2f'(x) + 4)\,dx = 2\int_{3}^{5} f'(x)\,dx + \int_{3}^{5} 4\,dx$
$= 2(f(5) - f(3)) + 4(5 - 3)$
$= 2\left(0 - \left(3 - \sqrt{5}\right)\right) + 8$
$= 2\left(-3 + \sqrt{5}\right) + 8 = 2 + 2\sqrt{5}$

— OR —

$\int_{3}^{5} (2f'(x) + 4)\,dx = [2f(x) + 4x]_{x=3}^{x=5}$
$= (2f(5) + 20) - (2f(3) + 12)$
$= (2 \cdot 0 + 20) - \left(2\left(3 - \sqrt{5}\right) + 12\right)$
$= 2 + 2\sqrt{5}$

2 : { 1 : Fundamental Theorem of Calculus
1 : answer

(c) $g'(x) = f(x) = 0 \Rightarrow x = -1,\ x = \frac{1}{2},\ x = 5$

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

On the interval $-2 \le x \le 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$.

3 : { 1 : $g'(x) = f(x)$
1 : identifies $x = -1$ as a candidate
1 : answer with justification

(d) $\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x} = \frac{10^1 - 3f'(1)}{f(1) - \arctan 1}$
$= \frac{10 - 3 \cdot 2}{1 - \arctan 1} = \frac{4}{1 - \frac{\pi}{4}}$

1 : answer

Question 4

(a) $V = \pi r^2 h = \pi (1)^2 h = \pi h$

$\left.\frac{dV}{dt}\right|_{h=4} = \pi \left.\frac{dh}{dt}\right|_{h=4} = \pi\left(-\frac{1}{10}\sqrt{4}\right) = -\frac{\pi}{5}$ cubic feet per second

$2: \begin{cases} 1: \frac{dV}{dt} = \pi \frac{dh}{dt} \\ 1: \text{answer with units} \end{cases}$

(b) $\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10}\sqrt{h}\right) = \frac{1}{200}$

Because $\frac{d^2h}{dt^2} = \frac{1}{200} > 0$ for $h > 0$, the rate of change of the height is increasing when the height of the water is 3 feet.

$3: \begin{cases} 1: \frac{d}{dh}\left(-\frac{1}{10}\sqrt{h}\right) = -\frac{1}{20\sqrt{h}} \\ 1: \frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt} \\ 1: \text{answer with explanation} \end{cases}$

(c) $\frac{dh}{\sqrt{h}} = -\frac{1}{10}\,dt$

$\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt$

$2\sqrt{h} = -\frac{1}{10}t + C$

$2\sqrt{5} = -\frac{1}{10} \cdot 0 + C \Rightarrow C = 2\sqrt{5}$

$2\sqrt{h} = -\frac{1}{10}t + 2\sqrt{5}$

$h(t) = \left(-\frac{1}{20}t + \sqrt{5}\right)^2$

$4: \begin{cases} 1: \text{separation of variables} \\ 1: \text{antiderivatives} \\ 1: \text{constant of integration and uses initial condition} \\ 1: h(t) \end{cases}$

Note: $0/4$ if no separation of variables

Note: max $2/4$ [1-1-0-0] if no constant of integration

Question 5

(a) $\int_0^2 (h(x) - g(x))\,dx = \int_0^2 \left(\left(6 - 2(x-1)^2\right) - \left(-2 + 3\cos\left(\frac{\pi}{2}x\right)\right)\right) dx$

$= \left[\left(6x - \frac{2}{3}(x-1)^3\right) - \left(-2x + \frac{6}{\pi}\sin\left(\frac{\pi}{2}x\right)\right)\right]_{x=0}^{x=2}$

$= \left(\left(12 - \frac{2}{3}\right) - (-4 + 0)\right) - \left(\left(0 + \frac{2}{3}\right) - (0 + 0)\right)$

$= 12 - \frac{2}{3} + 4 - \frac{2}{3} = \frac{44}{3}$

The area of R is $\frac{44}{3}$.

4 :
- 1 : integrand
- 1 : antiderivative of $3\cos\left(\frac{\pi}{2}x\right)$
- 1 : antiderivative of remaining terms
- 1 : answer

(b) $\int_0^2 A(x)\,dx = \int_0^2 \frac{1}{x+3}\,dx$

$= \left[\ln(x+3)\right]_{x=0}^{x=2} = \ln 5 - \ln 3$

The volume of the solid is $\ln 5 - \ln 3$.

2 :
- 1 : integral
- 1 : answer

(c) $\pi\int_0^2 \left((6 - g(x))^2 - (6 - h(x))^2\right) dx$

3 :
- 1 : limits and constant
- 1 : form of integrand
- 1 : integrand

Question 6

(a) $h'(2) = \frac{2}{3}$

1 : answer

(b) $a'(x) = 9x^2 h(x) + 3x^3 h'(x)$

$a'(2) = 9 \cdot 2^2 h(2) + 3 \cdot 2^3 h'(2) = 36 \cdot 4 + 24 \cdot \frac{2}{3} = 160$

3 : { 1 : form of product rule; 1 : $a'(x)$; 1 : $a'(2)$ }

(c) Because h is differentiable, h is continuous, so $\lim_{x\to 2} h(x) = h(2) = 4$.

Also, $\lim_{x\to 2} h(x) = \lim_{x\to 2} \frac{x^2 - 4}{1 - (f(x))^3}$, so $\lim_{x\to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4$.

Because $\lim_{x\to 2}\left(x^2 - 4\right) = 0$, we must also have $\lim_{x\to 2}\left(1 - (f(x))^3\right) = 0$.

Thus $\lim_{x\to 2} f(x) = 1$.

Because f is differentiable, f is continuous, so $f(2) = \lim_{x\to 2} f(x) = 1$.

Also, because f is twice differentiable, f' is continuous, so $\lim_{x\to 2} f'(x) = f'(2)$ exists.

Using L'Hospital's Rule,

$$\lim_{x\to 2} \frac{x^2 - 4}{1 - (f(x))^3} = \lim_{x\to 2} \frac{2x}{-3(f(x))^2 f'(x)} = \frac{4}{-3(1)^2 \cdot f'(2)} = 4.$$

Thus $f'(2) = -\frac{1}{3}$.

4 : { 1 : $\lim_{x\to 2} \frac{x^2 - 4}{1 - (f(x))^3} = 4$; 1 : $f(2)$; 1 : L'Hospital's Rule; 1 : $f'(2)$ }

(d) Because g and h are differentiable, g and h are continuous, so $\lim_{x\to 2} g(x) = g(2) = 4$ and $\lim_{x\to 2} h(x) = h(2) = 4$.

Because $g(x) \le k(x) \le h(x)$ for $1 < x < 3$, it follows from the squeeze theorem that $\lim_{x\to 2} k(x) = 4$.

Also, $4 = g(2) \le k(2) \le h(2) = 4$, so $k(2) = 4$.

Thus k is continuous at $x = 2$.

1 : continuous with justification